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1987 J. Phys. A: Math. Gen. 20 5397

(http://iopscience.iop.org/0305-4470/20/15/051)

J. Phys. A: Math. Gen. 20 (1987) 5397-5401. Printed in the UK

COMMENT

Isospectral Hamiltonians: generation of the soliton profile

C N Kumar

Institute of Physics, Sachivalaya Marg, Bhubaneswar, 751005, India

Received 31 March 1987

Abstract. The idea of generating a family of isospectral Hamiltonians from a given Hamiltonian using supersymmetric quantum mechanics is exploited in constructing the 'partner' stability equation for the ϕ^4 soliton stability equation. From the 'partner' stability equation, the soliton profile and the potential that admits the soliton solution are derived.

The generation of isospectral Hamiltonians using standard methods and via the techniques of supersymmetric quantum mechanics is currently receiving much attention. The general inequivalence of the two well established procedures of Abraham-Moses and Darboux was shown by Luban and Pursey (1986). The procedure of Abraham and Moses (1980) allows the construction of continuous families of isospectral Hamiltonians, i.e. new Hamiltonians with the same eigenvalue spectrum as that of the original Hamiltonian H. The Darboux procedure can be used to generate a one-parameter family of new Hamiltonians, all of which have the same eigenvalue spectrum as that of the original Hamiltonian except for (i) the addition of a new ground state whose energy is less than the ground-state energy of H or (ii) the deletion of the ground state from the spectrum of H. Both procedures are in general inequivalent except when one considers the case of the deletion of the ground state of the eigenvalue ε_0 of H and reintroducing a new ground state with the same energy (Luban and Pursey 1986). Pursey (1986a) constructed a third procedure of generating isospectral Hamiltonians which is different from the above mentioned procedures. The effect of this new procedure on reflection and transmission amplitudes and on norming constants for bound states has some interesting physical applications. Pursey (1986b), in a continuation of the series of papers, used isometric operators to provide a unified theory of these three procedures for generating one-parameter families of isospectral Hamiltonians. In another development, Sukumar (1985a) constructed families of isospectral Hamiltonians using supersymmetric quantum mechanics and discussed its connection with the procedure of Abraham and Moses. Nieto (1984) showed the relationship between supersymmetry and the inverse method in quantum mechanics. Mielnik (1984) use the factorisation method to generate an isospectral Hamiltonian for the harmonic oscillator problem.

The objective of the mathematical exercise of constructing a set of Hamiltonians which have the same eigenvalue spectrum as that of original Hamiltonian, with the exception of having an additional state or the deletion of ground state, from the original spectrum has interesting appplications. If the computational methods used for determining the eigenvalue spectrum converge less rapidly when applied to the original Hamiltonian than when applied to an isospectral partner, it may prove advantageous to compute the spectrum of eigenfunctions indirectly. In the context of quarkonium

0305-4470/87/155397+05\$02.50

physics the use of isospectral Hamiltonians permits us to modify the bound-state eigenfunctions of the system and therefore the values of wavefunction-dependent quantities, such as transition amplitudes, without destroying the agreement already achieved between the predicted and observed energy spectra (Pursey 1986b).

In this comment we present an application of the isospectral Hamiltonian approach in soliton physics. Solitons are solutions of non-linear field theories in $(1+1)$ dimensions which have finite extension and finite energy density. The stability of the soliton is ensured by the occurrence of a zero-energy ground state of the stability equation when small oscillations around the soliton are considered (Rajaraman 1982). Considering the stability equation as a one-dimensional Schrödinger-like equation for a particle in a potential $V(x)$, we can construct an isospectral partner for it. The partner stability equation will have the same energy spectrum as that of original equation. Then, as in Christ and Lee (1975), we 'generate' the soliton solution and hence the potential $V(\phi)$ which admits the soliton solution from the partner stability equation.

We consider the following field theory in $(1+1)$ dimensions (Rajaraman 1982):

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-V(\phi) \tag{1}$$

where

$$V(\phi)=\tfrac{1}{4}\lambda\phi^4-\tfrac{1}{2}m^2\phi^2. \tag{2}$$

At the classical level this model has two degenerate minima at $\phi=\pm m/\sqrt{\lambda}$. The field equation is

$$\partial^2\phi/\partial z^2-V'(\phi)=0 \tag{3}$$

where the prime denotes differentiation WRT ϕ. The soliton solution of this model is

$$\phi(z)=(m/\sqrt{\lambda})\tanh mz/\sqrt{2}. \tag{4}$$

The corresponding soliton mass is

$$M_s^{\mathrm{cl}}=\frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}. \tag{5}$$

The stability equation is

$$(-\nabla^2-m^2+3m^2\tanh^2 mz/\sqrt{2})\psi_i=\omega_i^2\psi_i. \tag{6}$$

Changing variables from z to $x=mz/\sqrt{2}$, this equation with $\varepsilon_i\equiv E_i^2=\omega_i^2/m^2$ becomes

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}+2-3\operatorname{sech}^2 x\right)\psi_i=\varepsilon_i\psi_i. \tag{7}$$

This is an exactly solvable problem and admits two bound states followed by scattering states. The bound-state solutions are

$$\varepsilon_0=0 \qquad \psi_0(x)=1/\cosh^2 x \tag{8}$$

$$\varepsilon_1=\tfrac{3}{2} \qquad \psi_1(x)=\sinh x/\cosh^2 x, \tag{9}$$

The knowledge of the ground-state wavefunction and its energy enables us to factorise equation (7). Once the factorisation is done as $H=A^+A^-+\varepsilon_0$, the energy spectrum of H and its partner $H_1 = A^-A^+ + \varepsilon_0$ are related via supersymmetry as $E_1^{(n)} = E^{(n+1)}(n = 0, 1, 2, \ldots)$ (Sukumar 1985b) where E, E_1 are the energy spectra of the Hamiltonians H and H_1 respectively. Equation (7) can be written as

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + 2 - 3\,\mathrm{sech}^2\,x \equiv A^+A^- + \varepsilon \tag{10}$$

where

$$A^{\pm} = (1/\sqrt{2})(\pm\partial/\partial x + \alpha(x))$$

and

$$\alpha(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_0 = -2\tanh x \qquad \varepsilon = \varepsilon_0.$$

With $\varepsilon_0 = 0$,

$$V(x) = \tfrac{1}{2}(\alpha^2(x) + \alpha'(x)) = 2 - 3\,\mathrm{sech}^2\,x.$$

The partner Hamiltonian has the form $H^1 = A^-A^+ + \varepsilon_0$ with

$$\begin{aligned} H^1 &= -\frac{1}{2}\frac{\partial^2}{\partial x^2} + 2 - \mathrm{sech}^2\,x \\ V^1(x) &= [\alpha^2(x) - \alpha'(x)]/2 = 2 - \mathrm{sech}^2\,x. \end{aligned} \tag{11}$$

This has only one bound state, $\varepsilon_0^1 = \frac{3}{2}$, as expected. At this stage, following Mielnik (1984), we ask whether the factorisation $H^1 = A^-A^+ + \varepsilon_0$ is unique or not. Consider that $H^1 = B^-B^+ + \varepsilon_0$ is another factorisation, where

$$B^{\pm} = (1/\sqrt{2})(\pm\partial/\partial x + \beta(x)).$$

An obvious particular solution is $\beta(x) = \alpha(x)$. Let the general solution be

$$\beta(x) = \alpha(x) + \phi(x).$$

This yields

$$\phi^2(x) + 2\phi(x)\alpha(x) - \phi'(x) = 0. \tag{12}$$

Introducing a new function $y = 1/\phi$, one ends up with a first-order linear inhomogeneous equation

$$y' + 2y\alpha + 1 = 0 \tag{13}$$

whose general solution is

$$y = \exp\int -2\alpha(x)\,\mathrm{d}x\left[-1\int\left(\exp\int 2\alpha(x)\,\mathrm{d}x\right)\mathrm{d}x + c\right] \tag{14}$$

where c is a constant.

With $\alpha(x) = (\mathrm{d}/\mathrm{d}x)\ln\psi_0$,

$$y = \frac{1}{\psi_0^2}\left(c - \int\psi_0^2\,\mathrm{d}x\right) \tag{15}$$

or

$$\phi(x) = \psi_0^2\left(c - \int\psi_0^2\,\mathrm{d}x\right)^{-1}. \tag{16}$$

Using $\psi_0=\operatorname{sech}^2 x$ for the ϕ^4 soliton which is under consideration, $\phi(x)$ becomes

$$\phi(x)=\frac{\operatorname{sech}^4 x}{(c-\tanh x+\frac{1}{3}\tanh^3 x)}. \tag{17}$$

As long as $|c|>\frac{2}{3}$, ϕ is non-singular. Hence we have another factorisation for H^1, i.e.

$$H^1=A^-A^++\varepsilon_0=B^-B^++\varepsilon_0. \tag{18}$$

At this stage the above equation offers little new information; however if we construct B^+B^- it is no longer H but a new Hamiltonian, H^N,

$$H^N=B^+B^-+\varepsilon_0=B^-B^+-[B^-,B^+]+\varepsilon_0$$
$$=H^1-[B^-,B^+].$$

Using $[B^-,B^+]=-\partial\beta/\partial x$

$$H^N=H^1+\partial\beta(x)/\partial x \qquad \text{or} \qquad V^N(x)=V^1(x)+\partial\beta(x)/\partial x. \tag{19}$$

H^N is the new Hamiltonian which can be viewed as the SUSY partner of H^1. The energy spectrum of H^N will have a new state in addition to H^1 energy spectrum. The identical part in both the spectra can be found from

$$H^NB^+=(B^+B^-+\varepsilon_0)B^+=B^+(B^-B^++\varepsilon_0)=B^+H^1. \tag{20}$$

The wavefunctions of H^N have correspondence with the wavefunction of H^1 through

$$\psi_1^N=B^+\psi_0,\ \psi_2^N=B^+\psi_1,\ldots,\ \psi_n^N=B^+\psi_{n-1}. \tag{21}$$

The missing eigenstate can be found from the relation

$$B^-\psi_0^N=0$$
$$(-\partial/\partial x+\beta(x))\psi_0^N=0 \tag{22}$$
$$\psi_0^N(x)=\exp\left(\int\beta(x)\,\mathrm{d}x\right)=\exp\left(\int(\alpha(x)+\phi(x))\,\mathrm{d}x\right).$$

Using $\alpha(x)=(\mathrm{d}/\mathrm{d}x)\ln\psi_0$ and equations (16) and (8)

$$\psi_0^N(x)=\psi_0(x)\left(c-\int\psi_0^2\,\mathrm{d}x\right)^{-1} \tag{23}$$

$$\psi_0^N(x)=\frac{\operatorname{sech}^2 x}{(c-\tanh x+\frac{1}{3}\tanh^3 x)} \qquad |c|>\tfrac{2}{3}. \tag{24}$$

By construction the energy of ψ_0^N is zero:

$$H^N\psi_0^N=(B^+B^-+\varepsilon_0)\psi_0^N=0 \qquad \text{since } \varepsilon_0=0. \tag{25}$$

We call H^N an isospectral partner of H in the sense of having the same energy spectrum as that of H. In a sense, non-uniqueness of factorising H has led us to construct one more parent Hamiltonian H^N in addition to the original Hamiltonian H.

Now that the partner stability equation has been constructed according to the method of Christ and Lee (1975) it is easy to generate the soliton solution from this equation. We use

$$\psi_0^N(x)=\mathrm{d}\phi^N(x)/\mathrm{d}x \tag{26}$$

to obtain the soliton solution $\phi_S^N(x)$ as a function of x, and the field theory is determined by

$$V(\phi)=\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2. \tag{27}$$

On integrating equation (26), using equation (24) with $c=1$,

$$\phi_S^N(x)=\frac{1}{2(K^2-1)}\ln\left(\frac{(Y-K)^2}{[Y^2+YK+(K^2-3)]}\right)$$
$$-\frac{3}{(K^2-1)[3(K^2-4)]^{1/2}}\tan^{-1}\left(\frac{(2Y+K)}{3(K^2+4)^{1/2}}\right) \tag{28}$$

where $Y=\tanh x$ and K is the real root of the cubic equation $K^3-3K+3=0$.

The algebraic complexity of expressing x in terms of ϕ in (28), and thus $V(\phi)(=\frac{1}{2}(d\phi/dx)^2)$ as a function of ϕ, hinders further study of the structure of the potential.

In the spirit mentioned at the beginning of this comment regarding the application of the 'partner' soliton, if the presence of soliton excitation in the system relies on the small oscillation 'data' and not on the structure of the soliton profile, the prescription described will be useful for generating a 'partner' soliton with the same data.

It is a pleasure to thank Dr A Khare for useful discussions.

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